An Alternating Direction Difference Scheme for Solving Four Dimension Reaction Diffusion Equation with Constant Coefficients

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Abstract. In the paper a compact alternate direct implicit difference method for the four dimensional constant coefficient reaction diffusion equations is studied. Firstly, a compact difference scheme is derived by combining methods of order reduction and dimension reduction. The expression of the truncation error is given in detail. Secondly, we introduce the transition layer variable methods which give tightly alternating direction difference scheme. The stability and convergence of difference schemes are proved by using Fourier method. The order of convergence is \(O(\tau^2 + h^4)\). Thirdly, Richardson’s extrapolation method is successfully applied the difference scheme and achieves the approximate solution with accuracy \(O(\tau^2 + h^4)\). Finally, the simulation results coincided with the theoretical analysis which verified the effectiveness of our method.

Keywords: Direction Difference, Reaction Diffusion Equation, Coefficients

1 Introduction

Reaction–diffusion equations have been used to describe the systems and problems in various domains of applied sciences. Many works have been devoted to the qualitative analysis of these systems (see PaoCV(1992)) and the numerical methods for computation of the solutions (GuY(2003), Liao(2002), PaoCV (1985), PaoCV (1995), PaoCV(1999), PaoCV(2002), Wang YM(2008) and Wang YM(2006)). In this paper, we present a numerical treatment of a system of four-dimensional reaction–diffusion equations with nonlinear reaction terms by a compact finite difference alternating direction implicit method. This includes the qualitative analysis of the resulting discrete system and a basic monotone iterative algorithm for the computation of numerical solutions. The reaction–diffusion system under consideration is given by:
Alternating direction implicit (ADI) methods are popular methods to solve two/three dimensional parabolic differential equations (see Dai W(2000), DouglasJJ(1964), Karaa S(2006), Peaceman(1995)). For this purpose, we only need to solve a sequence of tridiagonal systems. Hence, the overall computation is straightforward and fast. As a representative work, Liao(2002) presented a compact finite difference ADI method by using the Crank–Nicolson technique. Since an ADI technique is adopted, this method reduces the two-dimensional problem to two one-dimensional problems. This reduction gives a practical advantage in the computation of numerical solutions. However, its higher-order convergence was exhibited only numerically through two test examples in Liao(2002). To the best of our knowledge, there are no theoretical analysis (such as the existence–uniqueness problem and the convergence of numerical solutions) exist so far for this method. Therefore, it is necessary to develop a novel iterative algorithm to compute its solutions.

2 Construction and Truncation Error of the Compact Difference Scheme

It is assumed that the solution of (1) is sufficient smooth. $u(x,y,z,q,t) \in C^\infty(\Omega\times\{0,T\})$

If $v = \frac{\partial u}{\partial x}$, $w = \frac{\partial^2 u}{\partial y^2}$, $m = \frac{\partial^3 u}{\partial z^3}$, $g = \frac{\partial u}{\partial q}$, then (1) has equivalent equation

$$\frac{\partial u}{\partial t} = av + bw + cm + dg + f(x,y,z,q,t).$$

Equation (2) is first-order differential equations, which reduce the orders of equation by one comparing with (1). The scheme of equations (3) is one-dimensional problem and the dimensions drop three dimensions.

Based on equation (2) in $u(x,y,z,q,t^{n+\frac{1}{2}})$, then we have

$$\frac{\partial u}{\partial t}(x,y,z,q,t^{n+\frac{1}{2}}) = av(x,y,z,q,t^{n+\frac{1}{2}}) + bw(x,y,z,q,t^{n+\frac{1}{2}}) + cm(x,y,z,q,t^{n+\frac{1}{2}}) + dg(x,y,z,q,t^{n+\frac{1}{2}}) + f(x,y,z,q,t^{n+\frac{1}{2}}).$$

Due to the Taylor expansion, we can obtain

$$\delta U_{i,j,k,l}^{n+\frac{1}{2}} = aV_{i,j,k,l}^{n+\frac{1}{2}} + bW_{i,j,k,l}^{n+\frac{1}{2}} + cM_{i,j,k,l}^{n+\frac{1}{2}} + dG_{i,j,k,l}^{n+\frac{1}{2}} + f_{i,j,k,l}^{n+\frac{1}{2}} + r_{i,j,k,l}^{n+\frac{1}{2}} + o(\tau^4).$$
Let ABCD act on the top formula:

\[
ABCDU_{i,j,k,l}^{n+1} = aABCDV_{i,j,k,l}^{n+1} + bABCDW_{i,j,k,l}^{n+1} + cABCDM_{i,j,k,l}^{n+1} + dABCDG_{i,j,k,l}^{n+1}.
\]

\[+ ABCDf_{i,j,k,l}^{n+1} + o(\tau^4).
\]

(4)

ABCD\delta U_{i,j,k,l}^{n+1} = aABCD\delta U_{i,j,k,l}^{n+1} + bACD\delta U_{i,j,k,l}^{n+1} + cABD\delta U_{i,j,k,l}^{n+1} + dABC\delta U_{i,j,k,l}^{n+1} - \frac{\tau^2}{4} (ABCd \delta^2 \delta_t^2 + ACbd \delta^2 \delta_y^2 + ADbc \delta^2 \delta_z^2 + BCA\delta^2 \delta_r^2 + BDac \delta^2 \delta_q^2 + 
\]

\[CDab \delta^2 \delta_t^2 \delta U_{i,j,k,l}^{n+1} + \frac{\tau^2}{4} (abcD\delta^2 \delta_t^2 \delta_q^2 + bcad\delta^2 \delta_r^2 \delta_q^2 + abcdC\delta^2 \delta_y^2 \delta_q^2 + acbdB\delta^2 \delta_z^2 \delta_q^2) U_{i,j,k,l}^{n+1} + ABCDf_{i,j,k,l}^{n+1}.
\]

(5)

So difference scheme (5) attains highest approximation \(o(h^6)\) about the step length of space. It means this is Compact Difference Scheme.

3 Compact Alternate Direct Implicit Scheme

It is not difficult to see that (16) is equivalent to

\[
(A - \frac{1}{2} \alpha r^2 (u^{n+1} - u^n)) = [ABCd \tau \delta^2 + ACBr \tau \delta^2 + ACD \tau \delta^2 + BCD \tau \delta^2 + \frac{\tau^2}{4} (abcD\delta^2 \delta_t^2 \delta_q^2 + 
\]

\[bcad\delta^2 \delta_r^2 \delta_q^2 + abcdC\delta^2 \delta_y^2 \delta_q^2 + acbdB\delta^2 \delta_z^2 \delta_q^2) u^n + \tau ABCDf_{i,j,k,l}^{n+1}.
\]

(6)

Considering schemes (6), we know that the result used one-dimensional implicit scheme in the direction of \(x, y, z, q\). At each time level, this is a tridiagonal system of linear algebraic equations which can be solved. Therefore, less computation is involved.

4 The Stability and Convergence of Difference Scheme

According to Fourier method, suppose that \(u_{i,j,k,l}^n = \lambda^n e^{i\alpha x + i\beta y + i\gamma z + i\psi t}, (\varepsilon = \sqrt{\lambda})\). We obtain propagation factor of the scheme (5):

\[
\lambda = \left[ (1 - 2(\alpha r + 1/6)s_x)(1 - 2(\alpha r + 1/6)s_y)(1 - 2(\alpha r + 1/6)s_z)(1 - 2(\alpha r + 1/6)s_t) \right] / 
\]

\[\left[ (1 + 2(\alpha r - 1/6)s_x)(1 + 2(\alpha r - 1/6)s_y)(1 + 2(\alpha r - 1/6)s_z)(1 + 2(\alpha r - 1/6)s_t) \right].
\]
We have \(-1 \leq \frac{1 - 2(ar + 1/6)x_i}{1 + 2(ar - 1/6)x_i} \leq 1\). Hence \(|\lambda| \leq 1\).

According to Von Neumann condition, we know that the difference format (5) is unconditionally stable. Integrating with the discussion of section (2) and considering to stability and convergence of Lax theorem, the following theorem is obtained.

**Theorem 1** (13) is absolute stability and convergence order of the difference scheme is \(o(\tau^z + h^4)\).

5 Extrapolation Method

We can improve the convergence order of (13) by using extrapolation methods. According the discusses section (2), we have

\[
u(x, y, z, q, t) - \frac{16}{15} u_{i,j,k,l}^{(5)} \left( \frac{h}{2}, \frac{\tau}{4} \right) - \frac{1}{15} u_{i,j,k,l}^{(4)}(h, \tau) = o(\tau^4 + \tau^2 h^2 + h^4).
\]

This means \(\frac{16}{15} u_{i,j,k,l}^{(5)} \left( \frac{h}{2}, \frac{\tau}{4} \right) - \frac{1}{15} u_{i,j,k,l}^{(4)}(h, \tau)\) used as approximation of \(u(x, y, z, q, t)\), error order can be improved to \(o(\tau^4 + \tau^2 h^2 + h^4)\), for \(o(\tau^2 h^2) \leq \max \{o(\tau^4), o(h^4)\}\), so this error is order \(o(\tau^4 + h^4)\).

6 Numerical experiments

The initial value problem in \(D: [0 \leq x, y, z, q \leq 1, t \geq 0]\)

\[
\begin{align*}
\frac{\partial u}{\partial t} & = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
u \Big|_{x=0} & = \sin(x + y + z + q) \\
u \Big|_{x=1} & = e^{-4t} \sin(1 + y + z + q) \\
u \Big|_{y=0} & = e^{4t} \sin(x + z + q) \\
u \Big|_{y=1} & = e^{-4t} \sin(x + 1 + z + q) \\
u \Big|_{z=0} & = e^{4t} \sin(x + y + q) \\
u \Big|_{z=1} & = e^{-4t} \sin(x + y + 1 + q) \\
u \Big|_{q=0} & = e^{4t} \sin(x + y + z) \\
u \Big|_{q=1} & = e^{-4t} \sin(x + y + z + 1)
\end{align*}
\]

The exact solution of the above problem is \(u(x, y, z, q, t) = e^{-4t} \sin(x + y + z + q)\)

Taking \(h = 1/10, \tau = rh^2/100, r = 1/2, 1\), we compute at \(n = 200\) and compare with the exact solution. The following results in Table 1 are obtained.
Table 1. Comparison between result of each algorithm and exact value

<table>
<thead>
<tr>
<th>r</th>
<th>(j,k,l,q)</th>
<th>Exact solution</th>
<th>Scheme (16)</th>
<th>extrapolation</th>
</tr>
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<td>1/2</td>
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<td></td>
<td>(5,5,5,5)</td>
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<td>0.8267251579</td>
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<tr>
<td>1</td>
<td>(1,1,1,1)</td>
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<td>0.3219043133</td>
<td>0.3219042679</td>
</tr>
<tr>
<td></td>
<td>(3,3,3,3)</td>
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<td></td>
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<td>0.7516511241</td>
<td>0.7516510925</td>
</tr>
</tbody>
</table>

Table 1 shows that the scheme in this paper has a good fit with exact solution. Extrapolation methods in this paper have ten significant digits fit with exact solution, so it significantly improves the precision.

7 Conclusions

In this paper, we analyzed a compact alternate direct implicit difference method for solving a system of the four dimensional constant coefficient reaction diffusion equations. We obtained the existence and uniqueness of the numerical solution, and theoretically showed that the numerical solution has the accuracy of fourth-order in space and second-order in time. The numerical results presented coincide with the analysis very well and demonstrate the high efficiency of the method. We also extended the method of Richardson’s extrapolation method for four dimensional partial differential equations.

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References